q-Ehrhart polynomials of Gorenstein polytopes, Bernoulli umbra and related Dirichlet series

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Abstract

This article considers some q-analogues of classical results concerning the Ehrhart polynomials of Gorenstein polytopes, namely properties of their q-Ehrhart polynomial with respect to a good linear form. Another theme is a specific linear form Ψ (involving Carlitz' q-analogues of Bernoulli numbers) on the space of polynomials, for which one shows interesting behaviour on these q-Ehrhart polynomials. A third point is devoted to some related zeta-like functions associated with polynomials.

Introduction

This article deals with properties of the q-Ehrhart polynomial of Gorenstein lattice polytopes, with a specific linear form on polynomials and with its relationship to some Dirichlet series. But the initial motivation was something rather different. Let us start with a short account of this story.

It has proved useful and interesting to consider, as a kind of non-associative replacement of usual formal power series in one variable, some infinite formal sums of rooted trees with coefficients in a base ring, where rooted trees plays the role of the monomials x^n. These objects can then be multiplied and composed, making a setting very similar to the classical one. One could call them *tree-indexed series*.

Among all the tree-indexed series, there are two specific ones, playing a role similar to the usual exponential and logarithm power series. Let us call them A and Ω [7, 5]. Then A is an analog of the exponential and there is a very simple and nice formula for its coefficients which are all positive rationals. On the contrary, Ω is an analog of the logarithm and has very complicated rational coefficients with signs, some of them vanishing.

One intriguing problem is to understand for which trees does the coefficient in Ω vanish. Here the Ehrhart polynomials enters the game, as it is known that the coefficient Ω_T of a tree T in Ω can be expressed using the Ehrhart polynomial of a polytope attached to T.

After looking closely at the trees with vanishing coefficients, one observed that most of them (but not all) have a very special shape, namely all their leaves have the same height. It turns out that this shape implies that the associated polytope is Gorenstein. This was the starting point of the present article.

Once generalized as much as possible, the problem was then to prove that a specific linear form Ψ vanishes on the product of Ehrhart polynomials of two r-Gorenstein polytopes of odd total dimension.

It turns out that this vanishing property is best seen and explained when the classical notion of Ehrhart polynomial is replaced by the q-Ehrhart polynomial introduced in [6]. Instead of a vanishing property, one has to prove that a specific linear form takes values that are (up to sign) self-reciprocal elements of $\mathbb{Q}(q)$.

The vanishing properties that are obtained as a corollary when setting $q = 1$ are similar to the vanishing of the Bernoulli numbers of odd indices, which can be seen as a consequence of the functional equation of the Riemann ζ function. This inspired by analogy the study of a Dirichlet series attached to a polynomial, for which one describes the analytic continuation and obtains a simple expression of values at negative integers in term of the linear form Ψ.

Let us now describe the contents of this article.

In section 1, one states a simple symmetry property of the q-Ehrhart polynomials of Gorenstein polytopes.

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Sections 2 and 3 are about the general setting, sketching the landscape and gathering tools. Section 2 introduces a q-analog of the space of integer-valued polynomials and several subspaces and bases. In section 3, one introduces a linear operator Σ and two linear forms V and Ψ, compute some of their values and explain the relations of Ψ with some q-analogues of Bernoulli numbers due to Carlitz.

Section 4 contains the main results about q-Ehrhart polynomials. One first obtains a symmetry property for the coefficients of the q-Ehrhart polynomials of Gorenstein polytopes, when expressed in a particular basis. This statement is a q-analog of the well-known symmetry of the coefficients of the numerator of the Ehrhart series of Gorenstein polytopes. Using this symmetry, one then proves that the image by Ψ of the product of two Ehrhart polynomials of r-Gorenstein polytopes is (up to sign) self-reciprocal. A similar result is obtained in the special case of 1-Gorenstein (a.k.a. reflexive) polytopes.

In section 5, one lets $q = 1$ and goes back to the classical setting of Ehrhart polynomials. One first states the desired vanishing results, as easy consequences of the main results. This is illustrated by several examples. One also proposes a conjecture about what happens when one considers the powers of one fixed Gorenstein polytope. The sequence of rational numbers thus obtained seems to share properties with the Bernoulli numbers. This suggest to see them as the values at negative integers of a zeta-like function, by analogy with the classical relation between Bernoulli numbers and Riemann zeta function. This leads to the definition of a Dirichlet series attached to the Ehrhart polynomial.

In the last section 6, one proves that these Dirichlet series have a meromorphic continuation to $\mathbb{C}$ with just a simple pole at 1, and that their values at negative integers are given by the expected formula.

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0.1 Notations

Let us introduce some basic notations. The letter q will always stand for a formal parameter. It will either be considered as an element of $\mathbb{Q}(q)$ or as an element of $\mathbb{Z}[q, 1/q]$.

For $n \in \mathbb{N}$, we will denote by $[n]_q$ the q-integer $1 + q + \cdots + q^{n-1}$. Using instead the formula $\frac{q^n-1}{q-1}$, this can be extended to $n \in \mathbb{Z}$. One then has the obvious relations $[n]_{1/q} = q^{-n+1}[n]_q$ and $[-n]_q = -q^{-n}[n]_q$.

For $n \in \mathbb{N}$, we will denote by $[n]!_q$ the q-factorial of n, namely the product $[1]_q[2]_q \ldots [n]_q$.

For $m, n \in \mathbb{N}$, we will denote by $\begin{bmatrix} m \\ n \end{bmatrix}_q$ the q-binomial $\frac{[m]!_q}{[n]!_q[m-n]!_q}$.

For fixed $n \in \mathbb{N}$, this can be written as $([m]_q[m-1]_q \ldots [m-n+1]_q)/[n]!_q$, which makes sense for every $m \in \mathbb{Z}$. One then has the useful formulas $\begin{bmatrix} -m \\ n \end{bmatrix}_q = (-1)^n q^{-nm-\binom{n}{2}} \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q$ and $\begin{bmatrix} m \\ n \end{bmatrix}_{1/q} = q^{-m(n-m)} \begin{bmatrix} -m \\ n \end{bmatrix}_q$.

All the previous notations are rather standard for these q-analogues. One will need also some other notations, less classical.

For $n \in \mathbb{N}$, let $[n, x]_q$ be the polynomial $[n]_q + q^n x$. For $m, n \in \mathbb{N}$, let also define the polynomial

$$\begin{bmatrix} m, x \\ n \end{bmatrix}_q = \frac{[m-n+1, x]_q[m-n+2, x]_q \ldots [m, x]_q}{[n]_q!}.$$

When q is replaced by 1, they become $n + x$ and $\binom{m+x}{n}$.

These polynomials are defined in this way so that they have nice evaluations when x is replaced by a q-integer $[k]_q$. Indeed $[n, [k_q]]_q$ is just $[n+k]_q$ and therefore $\begin{bmatrix} m, [k]_q \\ n \end{bmatrix}_q = \begin{bmatrix} m+k \\ n \end{bmatrix}_q$.

They also satisfy the following translation properties:

$$[n, [k, x]_q]_q = [n+k, x]_q \quad \text{and} \quad \begin{bmatrix} m, [k,x]_q \\ n \end{bmatrix}_q = \begin{bmatrix} m+k, x \\ n \end{bmatrix}_q. \tag{1}$$

1 On q-Ehrhart polynomials of Gorenstein polytopes

One will use here some results of the article [6], where a q-analogue of the classical theory of Ehrhart polynomial has been introduced.

Recall that a lattice polytope P is called *reflexive* if it contains the lattice origin 0 and the dual polytope P^* is also a lattice polytope. These polytopes are used in the study of mirror symmetry in the setting of toric geometry. There is another closely related notion. A lattice polytope P is called r-*Gorenstein* (for some integer $r \geq 1$) if the dilated polytope rP is (up to lattice translation) reflexive.

For more on reflexive and Gorenstein polytopes, the reader can consult for example [2, 1, 13, 12, 14].

Let us now assume that P is an r-Gorenstein lattice polytope of dimension D. Let z_0 be the unique interior lattice point of rP.

Let λ be a linear form on the lattice, such that λ is positive on P and λ is not constant on any edge of P. These conditions are required for the definition of the q-Ehrhart polynomial (see [6] for details).

Then one can consider the q-Ehrhart polynomial $E_{P,\lambda}(x, q)$, defined by

$$E_{P,\lambda}([n]_q, q) = \sum_{s \in nP} q^{\lambda(s)}. \tag{2}$$

For short, it will be denoted by E when no ambiguity is possible. This is a polynomial in $\mathbb{Q}(q)[x]$.

Our first result is the following simple symmetry property.

Proposition 1.1 *The q-Ehrhart polynomial E satisfies*

$$E(x, q) = (-1)^D E(-q[r, x]_q, 1/q) q^{-\lambda(z_0)}. \tag{3}$$

This is a q-deformation of the classical relation

$$E(x) = (-1)^D E(-r - x) \tag{4}$$

for the Ehrhart polynomial of r-Gorenstein lattice polytopes.

Proof. Let $n \geq 1$ be an integer. By q-Ehrhart reciprocity [6, Th. 2.5], one knows that

$$E([-n]_q, q) = (-1)^D \sum_{s \in \mathrm{Int}(nP)} q^{-\lambda(s)}.$$

By the Gorenstein property, the translation by the vector z_0 gives an isomorphism of lattice polytopes from nP to $\mathrm{Int}((n + r)P)$. It follows that

$$E([-n - r]_q, q) = (-1)^D \sum_{s \in nP} q^{-\lambda(s+z_0)},$$

whose right hand side can be written as

$$(-1)^D E([n]_q, q)\Big|_{q=1/q} q^{-\lambda(z_0)} = (-1)^D E([n]_{1/q}, 1/q) q^{-\lambda(z_0)}.$$

Then consider the variables

$$x = [-n - r]_q, \quad X = [n]_{1/q}.$$

One can check that they are related by $X = -q[r, x]_q$. The statement follows. ■

Note that the product $P \times Q$ of two r-Gorenstein polytopes is still an r-Gorenstein polytope. Moreover, the q-Ehrhart polynomial of a product $P \times Q$ of polytopes, with respect to the linear form $\lambda \oplus \mu$, is the product $E_{P,\lambda} E_{Q,\mu}$. This is obviously compatible with the proposition.

If P is a polytope, let us call the **pyramid** over P the convex hull of $(0, 0)$ and $1 \times P$ in a lattice of one more dimension. The pyramid over an r-Gorenstein polytope is an $r + 1$-Gorenstein polytope.

2 Binomial bases for polynomials in x

Let us now consider the polynomial ring $\mathbb{Q}(q)[x]$ and some of its elements.

This ring has a basis over $\mathbb{Q}(q)$ given by the polynomials $\begin{bmatrix} n, x \\ n \end{bmatrix}_q$ for $n \geq 0$, which will be called the B-basis.

Let us now define A_q, as the subspace of $\mathbb{Q}(q)[x]$ generated over $\mathbb{Z}[q, 1/q]$ by the polynomials $\begin{bmatrix} n, x \\ n \end{bmatrix}_q$.

For an integer $d \in \mathbb{N}$, let us denote by $\mathsf{A}_q^{(d)}$ the subspace of A_q of polynomials of degree at most d.

Proposition 2.1 *The polynomials* $\left[{k,x \atop d}\right]_q$ *for* $k = 0, \dots, d$ *form a basis of* $\mathsf{A}_q^{(d)}$ *over* $\mathbb{Z}[q, 1/q]$.

Proof. This follows from lemma 2.2, which proves that the matrix of coefficients of these polynomials in the B-basis is triangular with powers of q on the diagonal. ∎

Lemma 2.2 *For integers* $0 \leq i \leq d$*, there holds*

$$\left[{i,x \atop d}\right]_q = \sum_{j=0}^{d} (-1)^{d-j} q^{-d(d-i)+\binom{d-j}{2}} \left[{d-i \atop d-j}\right]_q \left[{j,x \atop j}\right]_q. \tag{5}$$

Proof. It is enough to check this for all q-integers $[k]_q$. This becomes

$$\left[{i+k \atop d}\right]_q = \sum_{j=0}^{d} (-1)^{d-j} q^{-d(d-i)+\binom{d-j}{2}} \left[{d-i \atop d-j}\right]_q \left[{j+k \atop j}\right]_q.$$

This is an instance of the q-Chu-Vandermonde formula for the ${}_2\phi_1$ basic hypergeometric function, see for example [10, Appendix II, formula (II.7)]. ∎

Let us now describe the product in this basis.

Proposition 2.3 *For all integers* $0 \leq i \leq d$ *and* $0 \leq j \leq e$*, there holds*

$$\left[{i,x \atop d}\right]_q \left[{j,x \atop e}\right]_q = \sum_{0 \leq \ell \leq d+e} q^{(\ell-e-i)(\ell-d-j)} \left[{d+j-i \atop \ell-i}\right]_q \left[{e+i-j \atop \ell-j}\right]_q \left[{\ell,x \atop d+e}\right]_q. \tag{6}$$

Proof. As this is an equality of polynomials in x, it is enough to check that it holds for all positive q-integers, namely that

$$\left[{i+k \atop d}\right]_q \left[{j+k \atop e}\right]_q = \sum_{0 \leq \ell \leq d+e} q^{(\ell-e-i)(\ell-d-j)} \left[{d+j-i \atop \ell-i}\right]_q \left[{e+i-j \atop \ell-j}\right]_q \left[{\ell+k \atop d+e}\right]_q$$

holds for all $k \geq 0$. This equality is in fact an instance of the classical Pfaff-Saalschütz identity for the basic hypergeometric function ${}_3\phi_2$. It can be recovered for example by letting $d = -j, a = e+i, e = -i, b = d+j, c = -k-1$ in formula (4) of [15]. ∎

Proposition 2.1 and 2.3 together implies that the subspace A_q of $\mathbb{Q}(q)[x]$ is a commutative ring over $\mathbb{Z}[q, 1/q]$.

Let us now turn to a simple symmetry statement, for later use.

Proposition 2.4 *For all integers* d, r, k*, the polynomials* $\left[{k,x \atop d}\right]_q$ *have the following symmetry property:*

$$\left[{k,\, -q[r,x]_q \atop d}\right]_{1/q} = (-1)^d q^{\binom{d+1}{2}} \left[{r-1+d-k,\, x \atop d}\right]_q. \tag{7}$$

Proof. This is a simple computation using the definition of these polynomials. The left hand side is

$$\frac{[k-d+1, y]_{1/q} \dots [k, y]_{1/q}}{[1]_{1/q} \dots [d]_{1/q}}$$

with $y = -q[r, x]_q$. This can be rewritten as

$$\frac{q^{d-k}([k-d+1]_q - [r,x]_q) \dots q^{1-k}([k]_q - [r,x]_q)}{q^0 [1]_q \dots q^{-d+1}[d]_q}.$$

This becomes

$$(-1)^d q^{\binom{d}{2}} \frac{q(q^{r-1+d-k}x + [r-1+d-k]_q) \dots q(q^{r-k}x + [r-k]_q)}{[d]!_q},$$

which gives the expected result. ∎

3 Operator and linear forms

Let us define an endomorphism Σ of $\mathbb{Q}(q)[x]$ by

$$(\Sigma E)([n]_q) = \sum_{j=0}^{n} q^j E([j]_q), \tag{8}$$

for all polynomials E.

If E is the q-Ehrhart polynomial $E_{P,\lambda}$ of a polytope P and linear form λ, then ΣE is the q-Ehrhart polynomial of the pyramid over P as defined at the end of section 1, with the linear form $1 \oplus \lambda$.

Lemma 3.1 *For every integer $d \geq 0$, there holds*

$$\Sigma \begin{bmatrix} d,\, x \\ d \end{bmatrix}_q = \begin{bmatrix} d+1,\, x \\ d+1 \end{bmatrix}_q. \tag{9}$$

Proof. As an equality between polynomials in x, it is enough to check that it holds for every positive q-integer k. This becomes

$$\sum_{j=0}^{k} q^j \begin{bmatrix} d+j \\ d \end{bmatrix}_q = \begin{bmatrix} d+1+k \\ d+1 \end{bmatrix}_q.$$

This is a classical formula, which has a simple combinatorial proof using the description of q-binomials by paths in a rectangle according to their area. ■

Note that property (9) uniquely defines the linear operator Σ. This also proves that it acts on the subring A_q.

Lemma 3.2 *For all integers i and d with $0 \leq i \leq d$, there holds*

$$\Sigma \begin{bmatrix} i,\, x \\ d \end{bmatrix}_q = q^{d-i}\left(\begin{bmatrix} i+1,\, x \\ d+1 \end{bmatrix}_q - \begin{bmatrix} i \\ d+1 \end{bmatrix}_q\right). \tag{10}$$

Proof. To prove this equality of polynomials in x, it is enough to check the statement for every positive q-integer $[k]_q$. This becomes

$$\sum_{j=0}^{k} q^j \begin{bmatrix} i+j \\ d \end{bmatrix}_q = q^{d-i}\left(\begin{bmatrix} i+1+k \\ d+1 \end{bmatrix}_q - \begin{bmatrix} i \\ d+1 \end{bmatrix}_q\right).$$

This holds because

$$\sum_{j=0}^{k} q^{j-d} \begin{bmatrix} j \\ d \end{bmatrix}_q = \begin{bmatrix} k+1 \\ d+1 \end{bmatrix}_q,$$

which is a classical formula, equivalent to (3). ■

Let us define next a linear form V from A_q to $\mathbb{Q}(q)$ by

$$V(E) = \lim_{x \to [-1]_q} \frac{E(x) - E([-1]_q)}{1 + qx}. \tag{11}$$

Note that $[-1]_q = -1/q$. This operator is therefore essentially the derivative of E at $x = [-1]_q$, up to a multiplicative factor of q.

Let us now define another linear form Ψ on A_q by the composition

$$\Psi(E) = V\Sigma E. \tag{12}$$

One will later study the values of the linear form Ψ on the q-Ehrhart polynomials of Gorenstein polytopes.

Let us first compute the values of Ψ on the basis elements.

Proposition 3.3 *For all integers $0 \leq i \leq d$, there holds*

$$\Psi(\left[\begin{smallmatrix} i,\, x \\ d \end{smallmatrix}\right]_q) = \frac{(-1)^{d-i} q^{-\binom{d-i}{2}}}{[d+1]_q \left[\begin{smallmatrix} d \\ i \end{smallmatrix}\right]_q}. \tag{13}$$

Proof. Using formula (10), one can compute

$$\Psi \left[\begin{smallmatrix} i,\, x \\ d \end{smallmatrix}\right]_q = q^{d-i} V \left(\left[\begin{smallmatrix} i+1,\, x \\ d+1 \end{smallmatrix}\right]_q - \left[\begin{smallmatrix} i \\ d+1 \end{smallmatrix}\right]_q \right) = q^{d-i} V \left(\frac{[i-d+1, x]_q \dots [i+1, x]_q}{[d+1]!_q} - \frac{[i-d]_q \dots [i]_q}{[d+1]!_q} \right).$$

By definition, the operator V is proportional to the derivative at $[-1]_q$. This implies that one gets

$$q^{d-i} \frac{([i-d]_q \dots [-1]_q)([1]_q \dots [i]_q)}{[d+1]!_q},$$

which can be readily rewritten as the expected result. ■

From these values, one deduces the following lemma.

Lemma 3.4 *For every polynomial $E \in \mathbb{Q}(q)[x]$, there holds*

$$q\Psi(E(1+qx)) - \Psi(E) = (q-1)E(0) + \partial_x E(0). \tag{14}$$

Proof. As both sides are linear in E, it is enough to check this identity for every basis element $E = \left[\begin{smallmatrix} d,\, x \\ d \end{smallmatrix}\right]_q$. First note that $\Psi(E) = \frac{1}{[d+1]_q}$ by proposition 3.3. Then using (10), one computes

$$q\Psi(E(1+qx)) = qV\Sigma(E(1+qx)) = V(\left[\begin{smallmatrix} d+2,\, x \\ d+1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} d+1 \\ d+1 \end{smallmatrix}\right]) = V(\left[\begin{smallmatrix} d+2,\, x \\ d+1 \end{smallmatrix}\right]).$$

By a direct computation using that V is proportional to the derivative at $[-1]_q$, this is $\sum_{j=1}^{d+1} \frac{q^j}{[j]_q}$. The same computation gives that $\partial_x E(0) = \sum_{j=1}^{d} \frac{q^j}{[j]_q}$. The result follows. ■

Let us now introduce the q-Bernoulli numbers of Carlitz by the formula

$$\Psi(x^n) = B_{q,n}, \tag{15}$$

for $n \geq 0$.

These rationals fractions, introduced by Carlitz in [4], are q-analogues of the Bernoulli numbers with nice properties. In particular, they only have simples poles at non-trivial roots of unity, and their value at $q = 1$ are the classical Bernoulli numbers. To see that (15) gives the same definition as Carlitz one, one can use lemma 3.4 applied to the monomials x^n.

Let us now go back to the study of Ψ. One will need the following result later.

Proposition 3.5 *For all integers $0 \leq i \leq d$ and $0 \leq j \leq e$, there holds*

$$\Psi(\left[\begin{smallmatrix} i,\, x \\ d \end{smallmatrix}\right]_q \left[\begin{smallmatrix} j,\, x \\ e \end{smallmatrix}\right]_q) = \frac{(-1)^{d-i+e-j} q^{-\binom{d-i}{2}+(d-i)(e-j)-\binom{e-j}{2}}}{[d+e+1]_q \left[\begin{smallmatrix} d+e \\ d-i+j \end{smallmatrix}\right]_q}. \tag{16}$$

Proof. Let us compute $\Psi(\left[\begin{smallmatrix} i,\, x \\ d \end{smallmatrix}\right]_q \left[\begin{smallmatrix} j,\, x \\ e \end{smallmatrix}\right]_q)$. By proposition 2.3, this is

$$\sum_{0 \leq \ell \leq d+e} q^{(\ell-e-i)(\ell-d-j)} \left[\begin{smallmatrix} d+j-i \\ \ell-i \end{smallmatrix}\right]_q \left[\begin{smallmatrix} e+i-j \\ \ell-j \end{smallmatrix}\right]_q \Psi(\left[\begin{smallmatrix} \ell,\, x \\ d+e \end{smallmatrix}\right]_q).$$

By proposition 3.3, this is

$$\sum_{0 \leq \ell \leq d+e} q^{(\ell-e-i)(\ell-d-j)} \left[\begin{smallmatrix} d+j-i \\ \ell-i \end{smallmatrix}\right]_q \left[\begin{smallmatrix} e+i-j \\ \ell-j \end{smallmatrix}\right]_q \frac{(-1)^{d+e-\ell} q^{-\binom{d+e-\ell}{2}}}{[d+e+1]_q \left[\begin{smallmatrix} d+e \\ \ell \end{smallmatrix}\right]}.$$

Using lemma 3.6, this becomes the expected result. ■

Lemma 3.6 *Let $0 \leq i \leq d$ and $0 \leq j \leq e$ be integers. Then*

$$\sum_{0 \leq \ell \leq d+e} (-1)^{\ell} q^{(\ell-e-i)(\ell-d-j)-\binom{d+e-\ell}{2}} \frac{\binom{e}{\ell-i}_q \binom{d}{\ell-j}_q}{\binom{d+e}{\ell}_q} = \frac{(-1)^{i+j} q^{-\binom{d-i}{2}+(d-i)(e-j)-\binom{e-j}{2}}}{\binom{d+e}{d}_q}.$$

Proof. One can assume without loss of generality that $i \geq j$. This can then be reformulated as an hypergeometric identity for the function ${}_3\phi_2$. This formula can be deduced from [10, Appendix III, formula (III.10)]. ■

4 Symmetry of coefficients and self-reciprocal values

Let P be an r-Gorenstein lattice polytope of dimension D. Let $E(x, q)$ be its q-Ehrhart polynomial with respect to a linear form λ.

Let d be the degree of $E(x, q)$. Using proposition 2.1, let us write $E(x, q)$ as follows:

$$E(x, q) = \sum_{j=0}^{d} c_j \left[\begin{smallmatrix} j,\, x \\ d \end{smallmatrix}\right], \tag{17}$$

for some coefficients c_j in $\mathbb{Q}(q)$.

Proposition 4.1 *The coefficients c_k vanish for $0 \leq k \leq r-2$. Moreover*

$$c_k = (-1)^{D+d} q^{\binom{d+1}{2}-\lambda(z_0)} c_{r-1+d-k}(1/q). \tag{18}$$

Proof. Because P is an r-Gorenstein polytope, the dilated polytopes kP have an empty interior if $1 \leq k \leq r-1$. This implies that $E(x, q)$ vanishes at the q-integers $[-1]_q, \ldots, [1-r]_q$. This in turn implies the vanishing of the coefficients $c_0, \ldots, c_{r-2}$ (by an easy induction).

Let us now show that the symmetry property of proposition 2.4 together with the symmetry property of proposition 1.1 implies the expected symmetry of the coefficients. One computes

$$\begin{aligned}
(-1)^D E(-q[r, x]_q, 1/q) q^{-\lambda(z_0)} &= (-1)^D \sum_{k=r-1}^{d} c_k(1/q) \left[\begin{smallmatrix} k,\, -q[r,x]_q \\ d \end{smallmatrix}\right]_{1/q} q^{-\lambda(z_0)} \\
&= (-1)^D \sum_{k=r-1}^{d} c_k(1/q) (-1)^d q^{\binom{d+1}{2}} \left[\begin{smallmatrix} r-1+d-k,\, x \\ d \end{smallmatrix}\right]_q q^{-\lambda(z_0)} \\
&= (-1)^D \sum_{k=r-1}^{d} c_{r-1+d-k}(1/q) (-1)^d q^{\binom{d+1}{2}} \left[\begin{smallmatrix} k,\, x \\ d \end{smallmatrix}\right]_q q^{-\lambda(z_0)}.
\end{aligned} \tag{19}$$

One then identifies the coefficients with (17) to get the expected equality. ■

This statement is a q-analogue of the usual symmetry $c_k = (-1)^{D+d} c_{r-1+d-k}$ for r-Gorenstein polytopes. In the classical setting, the numbers c_k are the coefficients of the numerator of the Ehrhart series (h-vector).

Let now $r \geq 1$ be a fixed integer. Let P and Q be two r-Gorenstein lattice polytopes of dimensions D and E. Let E_P and E_Q be their q-Ehrhart polynomials with respect to some linear forms λ and μ (omitted to keep the notation short). Let d and e be the degrees of these polynomials.

Let us introduce the shortcuts $Z = \lambda(z_0)$ and $Z' = \mu(z_0')$ where z_0 and z_0' are the unique interior points in the dilated polytopes rP and rQ. Let us write

$$E_p = \sum_{0 \leq i \leq d} c_i \left[\begin{smallmatrix} i,\, x \\ d \end{smallmatrix}\right]_q \quad \text{and} \quad E_Q = \sum_{0 \leq j \leq e} c'_j \left[\begin{smallmatrix} j,\, x \\ e \end{smallmatrix}\right]_q. \tag{20}$$

Let s_{-k} be the shift (with offset $-k$) defined by $\mathsf{s}_{-k}(P)([n]_q) = P([n-k]_q)$ for all $n \in \mathbb{Z}$ or equivalently by $(\mathsf{s}_{-k}P)(x) = P([-k, x])$. Then one has

$$\mathsf{s}_{-k}E_p = \sum_{0 \leq i \leq d} c_{i+k} \begin{bmatrix} i, x \\ d \end{bmatrix}_q \quad \text{and} \quad \mathsf{s}_{-k}E_Q = \sum_{0 \leq j \leq e} c'_{j+k} \begin{bmatrix} j, x \\ e \end{bmatrix}_q, \tag{21}$$

for all $0 \leq k \leq r-1$ (using the vanishing statement in proposition 4.1).

Theorem 4.2 *For all $0 \leq k \leq r-1$, the value at $1/q$ of the fraction $q^{-k}\Psi(\mathsf{s}_{-k}(E_P E_Q))$ is $(-1)^{D+E} q^{Z+Z'+r-1}$ times itself.*

Proof. Let us first compute $\Psi(\mathsf{s}_{-k}(E_P E_Q))$ using the expressions (20) and proposition 3.5. One gets

$$\frac{(-1)^{d+e}}{[d+e+1]_q} \sum_{i,j} (-1)^{i+j} c_{i+k} c'_{j+k} \frac{q^{-\binom{d-i}{2} + (d-i)(e-j) - \binom{e-j}{2}}}{\begin{bmatrix} d+e \\ d-i+j \end{bmatrix}_q}.$$

Let us now replace q by $1/q$ in this expression. One gets

$$\frac{(-1)^{d+e}}{[d+e+1]_{1/q}} \sum_{i,j} (-1)^{i+j} c_{i+k}(1/q) c'_{j+k}(1/q) \frac{q^{\binom{d-i}{2} - (d-i)(e-j) + \binom{e-j}{2}}}{\begin{bmatrix} d+e \\ d-i+j \end{bmatrix}_{1/q}}.$$

Using (18) once for P and once for Q, this becomes

$$\frac{(-1)^{D+E} q^{d+e}}{[d+e+1]_q} \sum_{i,j} (-1)^{i+j} c_{r-1+d-i-k} c'_{r-1+e-j-k} \frac{q^{\binom{d-i}{2} - (d-i)(e-j) + \binom{e-j}{2} - \binom{d+1}{2} + Z - \binom{e+1}{2} + Z' + (d-i+j)(e-j+i)}}{\begin{bmatrix} d+e \\ d-i+j \end{bmatrix}_q}.$$

Changing the indices of summations $i \longleftrightarrow r-1+d-i-2k$ and $j \longleftrightarrow r-1+e-j-2k$, one gets (after simplifications in the powers of q)

$$\frac{q^{-2k} q^{Z+Z'+r-1} (-1)^{D+E}}{[d+e+1]_q} \sum_{i,j} (-1)^{d-i+e-j} c_{i+k} c'_{j+k} \frac{q^{-\binom{d-i}{2} + (d-i)(e-j) - \binom{e-j}{2}}}{\begin{bmatrix} d+e \\ e+i-j \end{bmatrix}_q}.$$

Up to the power of q in front of the sum, this is $(-1)^{D+E}$ times the initial expression for $\Psi(\mathsf{s}_{-k}(E_P E_Q))$. ∎

In the special case of reflexive (*i.e.* 1-Gorenstein) polytopes, it is not necessary to consider a product of two polytopes to obtain a similar result. Let us now assume that P is reflexive.

Theorem 4.3 *The value at $1/q$ of the fraction $\Psi(E_P)$ is $(-1)^D q^Z$ times itself.*

Proof. Let us first compute $\Psi(E_P)$ using the expression (20) for P and proposition 3.3. One gets

$$\frac{(-1)^d}{[d+1]_q} \sum_i (-1)^i c_i \frac{q^{-\binom{d-i}{2}}}{\begin{bmatrix} d \\ i \end{bmatrix}_q}.$$

Let us now replace q by $1/q$ in this expression. One gets

$$\frac{(-1)^d}{[d+1]_{1/q}} \sum_i (-1)^i c_i(1/q) \frac{q^{\binom{d-i}{2}}}{\begin{bmatrix} d \\ i \end{bmatrix}_{1/q}}.$$

Using (18) for P (and the hypothesis $r = 1$), this becomes

$$\frac{(-1)^D q^d}{[d+1]_q} \sum_i (-1)^i c_{d-i} \frac{q^{\binom{d-i}{2} - \binom{d+1}{2} + Z + i(d-i)}}{\begin{bmatrix} d \\ i \end{bmatrix}_q}.$$

Changing the index of summation $i \longleftrightarrow d-i$, one gets (after simplifications in the powers of q)

$$\frac{q^Z (-1)^D}{[d+1]_q} \sum_i (-1)^{d-i} c_i \frac{q^{-\binom{d-i}{2}}}{\begin{bmatrix} d \\ i \end{bmatrix}_q}.$$

Up to the power q^Z in front of the sum, this is $(-1)^D$ times the initial expression for $\Psi(E_P)$. ∎

In fact, the self-reciprocal fractions involved in theorem 4.2 are all the same. Keeping the same notations, one has the following result.

Proposition 4.4 *The fractions $q^{-k}\Psi(\mathsf{s}_{-k}(E_P E_Q))$ for $k = 0, 1, \ldots, r-1$ are all equal.*

Proof. Let us apply lemma 3.4 to the polynomial $\mathsf{s}_{-k}(E_P E_Q)$ for $k = 1, \ldots, r-1$. One gets

$$\begin{aligned} &q\Psi(\mathsf{s}_{1-k}(E_P E_Q)) - \Psi(\mathsf{s}_{-k}(E_P E_Q)) \\ &\qquad = (q-1)E_P([-k]_q)E_Q([-k]_q) + q^{-k}\partial_x E_P([-k]_q)E_Q([-k]_q) + q^{-k}E_P([-k]_q)\partial_x E_Q([-k]_q). \end{aligned}$$

Because of the r-Gorenstein property, the right hand side vanishes for $k = 1, \ldots, r-1$. This implies the statement. ■

5 Classical case $q = 1$

One can state purely classical corollaries of theorems 4.2 and 4.3 by letting $q = 1$. The q-Ehrhart polynomial becomes the Ehrhart polynomial, and does no longer depend on the choice of a linear form λ.

In this context, Ψ becomes the linear form on the space $\mathbb{Q}[x]$ that maps x^n to the Bernoulli number B_n. The operator s_{-k} becomes the evaluation of polynomials in x at $x-k$. One obtains the following statements.

Theorem 5.1 *Let P be a product of at least two r-Gorenstein polytopes. Let E_P be its Ehrhart polynomial. The numbers $\Psi(\mathsf{s}_{-k}E_P)$ for $k = 0, 1, \ldots, r-1$ are all equal. If moreover the dimension of P is odd, then they all vanish.*

Theorem 5.2 *Let P be a reflexive polytope. Let E_P be its Ehrhart polynomial. If the dimension of P is odd, then $\Psi(E_P) = 0$.*

Let us now consider some simple examples.

Let P be the polytope with vertices 0 and 1 in $\mathbb{Z}$. This is a 2-Gorenstein polytope, with Ehrhart polynomial $x+1$. One deduces from theorem 5.1 that $\Psi((x+1)^n) = 0$ for all odd $n \geq 3$. By definition of the Bernoulli numbers, the expression $\Psi((x+1)^n)$ is just B_n itself, which is well-known to vanish in this case.

Let now P be the polytope with vertices 0 and 2 in $\mathbb{Z}$. This is a reflexive polytope (up to translation), with Ehrhart polynomial $1 + 2x$. Therefore theorem 5.2 implies that $\Psi(1 + 2x) = 0$, which is indeed the case because $B_0 = 1$ and $B_1 = -1/2$.

Let us consider a more complicated example. There exists a reflexive simplex in dimension 5 with 355785 lattice points [13]. Its Ehrhart polynomial is $E = 271803x^5/5 + 271803x^4/2 + 118594x^3 + 83979x^2/2 + 24692x/5 + 1$. One can check directly that its image by Ψ vanishes, as well as the images by Ψ of its small odd powers. By contrast, the even values do not vanish, for example $\Psi(E^2) = -48827203879/165$.

As an interesting counter-example, consider the triangle in $\mathbb{Z}^2$ with vertices $(0,0)$, $(1,0)$ and $(1,1)$. The Ehrhart polynomial is $E = \binom{x+2}{2}$. The first few values of $\Psi(E^i)$ are given by

$$1, 1/3, 1/30, -1/105, 1/210, -1/231, 191/30030, -29/2145, 2833/72930, \ldots \tag{22}$$

There is no vanishing here, as this polytope is 3-Gorenstein, but not a product of two such polytopes. One can note that these coefficients have appeared in the work of Ramanujan, in an asymptotic formula involving triangular numbers (see number (9) of [3, Chapter 38]).

5.1 Bernoulli-like numbers attached to Gorenstein polytopes

Let P be an r-Gorenstein polytope of odd dimension D and let E_P be its Ehrhart polynomial. As a special case of theorem 5.1, the rationals numbers $\Psi(E_P^k)$ attached to the powers E_P^k vanish for every odd integer $k \geq 3$. This can be seen as an analog of the same statement for Bernoulli numbers.

This suggest, for any fixed r-Gorenstein polytope P, to think about the sequence $\Psi(E_p^k)_{k\geq 0}$ as some kind of Bernoulli-like numbers attached to the Gorenstein polytope P.

It seems that at least one other property of Bernoulli numbers extend to the Bernoulli-like numbers, namely the following alternating sign property, which is is well-known for the Bernoulli numbers.

Conjecture 5.3 *If the dimension of P is odd, the signs of the non-zero $\Psi(E_P^k)$ alternate.*

For example, consider the 3-dimensional simplex with vertices $(0,0,0),(1,0,0),(1,1,0)$ and $(1,1,1)$. Its Ehrhart polynomial is $E=\binom{x+3}{3}$. The first few values of $\Psi(E^k)$ are

$$1, 1/4, 1/140, 0, -41/60060, 0, 50497/19399380, 0, -13687983/148728580, 0, 485057494433/30855460020, \dots$$

In the case of even dimension, it seems also that the signs are alternating, see for example (22).

These alternating conjectures are related to the behaviour of the number of zeroes in $[0,1]$ of the q-analogues of these numbers and to the next topic, namely continuous interpolation of the Bernoulli-like number by zeta-like functions.

5.2 Zeta functions of polynomials

Given a polynomial E in $\mathbb{Q}[x]$ taking positive values on $\mathbb{N}$, one can consider a kind of zeta function attached to E, defined by

$$Z(E;s)=\sum_{n\geq 0}\frac{\partial_x E(n)}{E(n)^s}, \tag{23}$$

for complex numbers s with $\Re(s)>1$.

One will be mostly interested in the case where E is the Ehrhart polynomial of a lattice polytope P. For example, one gets in this way

$$\sum_{n\geq 0}\frac{1}{(1+n)^s}=\zeta(s) \quad \text{and} \quad \sum_{n\geq 0}\frac{2}{(1+2n)^s}=2(1-2^{-s})\zeta(s) \tag{24}$$

for the two Gorenstein polytopes of dimension 1.

One shows in the next section that under some mild hypotheses on E the function $Z(E;s)$ is a meromorphic function of the complex parameter s with only a single pole at 1 with residue 1. Moreover its values at negative integers are given in terms of Ψ and E by the formula

$$Z(E;1-k)=-\frac{\Psi(E^k)}{k} \tag{25}$$

for all $k\in\mathbb{N}^*$.

Before proving this in the next section, let us give an heuristic argument. By letting $q=1$ in lemma 3.4, one gets

$$\Psi(E(1+x))-\Psi(E)=\partial_x E(0).$$

After a telescoping summation, one gets

$$\Psi(E(\ell+x))-\Psi(E)=\sum_{j=0}^{\ell-1}\partial_x E(j)$$

For polynomials that are powers (of the shape F^k for some F), one therefore gets

$$\Psi(F(\ell+x)^k)-\Psi(F^k)=k\sum_{j=0}^{\ell-1}\partial_x F(j)F(j)^{k-1}. \tag{26}$$

Formally going to the limit $\ell=\infty$ (and assuming that the first term of the left-hand side disappears) gives formula (25).

6 Study of zeta-like functions

Let $(B_k)_{k\geq 0}$ be the sequence of Bernoulli numbers. Recall the linear operator $\Psi : \mathbb{C}[X] \to \mathbb{C}$ defined by

$$\Psi(X^k) = B_k \quad \forall k \in \mathbb{N}.$$

By the usual properties of Bernoulli numbers, there also holds

$$\Psi((X+1)^k) = (-1)^k \ B_k \quad \forall k \in \mathbb{N}. \tag{27}$$

For all $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ and all $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we will use in the sequel the following notations: $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \dots x_d^{\alpha_d}$ and $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_d$. One denotes also for any $z \in \mathbb{C}$ verifying $\Re z > 0$ and any $s \in \mathbb{C}$, $z^s = e^{s \log x}$ where log is the principal determination of the logarithm.

The purpose of this section is to prove the following result:

Theorem 6.1 *Let $E \in \mathbb{R}[X]$ be a polynomial of degree $d \geq 1$. Let $a_1, \dots, a_d \in \mathbb{C}$ be the roots (not necessarily distinct) of E. Let $A \in \mathbb{N}^*$ such $\forall x \geq A \ \Re E(x) > 0$. One considers the Dirichlet series*

$$Z_A(E; s) := \sum_{n=A}^{+\infty} \frac{E'(n)}{E(n)^s}.$$

Then:

1. *$s \mapsto Z_A(E; s)$ converges absolutely in the half-plane $\{\Re(s) > 1\}$ and has a meromorphic continuation to the whole complex plane $\mathbb{C}$;*
2. *the meromorphic continuation of $Z_A(E; s)$ has only one simple pole in $s = 1$ with residue 1.*
3. *for any $M \in \mathbb{N}^*$, $\quad Z_A(E; 1-M) = -\frac{1}{M}\Psi(E(X+1)^M) - \sum_{n=1}^{A-1} E(n)^{M-1} E'(n)$.*

Remark 6.2 *By taking $A = 1$ and using the shifted polynomial $E(X-1)$ in the previous theorem, point 3 gives the formula (25). Note that summation in $Z_1(E(X-1); s)$ starts at 1, whereas summation in (23) starts at 0.*

Remark 6.3 *Point 1 of the theorem 6.1 is classic, even in a very general framework (see for example [11] or [9]). Our method in this paper is simple and provides, in addition to point 1, the new points 2 and 3 above. In [8] an analogue of point 3 was obtained for twisted Dirichlet series. However, the method of [8] uses the* **holomorphy** *of twisted Dirichlet series in the whole space and therefore can not be used in our setting here.*

One needs the following elementary lemma:

Lemma 6.4 *Let $d \in \mathbb{N}^*$ and $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{N}^d \setminus \{(0, \dots, 0)\}$. Set $\delta = (2 \max_j |a_j|)^{-1} > 0$. Then, for any $N \in \mathbb{N}$, any $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{C}^d$ and any $x \in [-\delta, \delta]$, we have*

$$\prod_{j=1}^{d} (1 - xa_j)^{-s_j} = \sum_{\ell=0}^{N} c_\ell(\mathbf{s}) \ x^\ell + x^{N+1} \rho_N(x; \mathbf{s}) \tag{28}$$

where

$$c_\ell(\mathbf{s}) = (-1)^\ell \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^d \\ |\boldsymbol{\alpha}| = \ell}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{j=1}^{d} \binom{-s_j}{\alpha_j}$$

and

$$\rho_N(x; \mathbf{s}) = (-1)^{N+1}(N+1) \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^d \\ |\boldsymbol{\alpha}| = N+1}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{j=1}^{d} \binom{-s_j}{\alpha_j} \int_0^1 (1-t)^N \prod_{j=1}^{d} (1 - txa_j)^{-s_j - \alpha_j} \ dt.$$

Moreover we have:

1. *for any $x \in [-\delta, \delta]$, $\mathbf{s} \mapsto \rho_N(\mathbf{s}; x)$ is holomorphic in the whole space $\mathbb{C}^d$;*
2. *for any compact subset K of $\mathbb{C}^d$, there exists a constant $C = C(K, \mathbf{a}, N, d) > 0$ such that*

$$\forall (\mathbf{s}, x) \in K \times [-\delta, \delta] \qquad |\rho_N(\mathbf{s}; x)| \leq C.$$

Proof. Let us fix $\mathbf{s} \in \mathbb{C}^d$. One considers the function ϕ defined in $[-\delta, \delta]$ by $\phi(x) = \prod_{j=1}^d (1 - xa_j)^{-s_j}$. The function ϕ is infinitely differentiable in $[-\delta, \delta]$ and an induction on ℓ shows that for all $\ell \in \mathbb{N}$ and all $x \in [-\delta, \delta]$,

$$\frac{\phi^{(\ell)}(x)}{\ell!} = (-1)^\ell \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^d \\ |\boldsymbol{\alpha}| = \ell}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{j=1}^d \binom{-s_j}{\alpha_j} \prod_{j=1}^d (1 - xa_j)^{-s_j - \alpha_j}.$$

The identity (28) then follows from the application of the Taylor formula with integral remainder at $x = 0$.

The second part of the lemma follows from the theorem of holomorphy under the integral sign. This completes the proof of Lemma 6.4. ■

Proof. (points 1 and 2 of Theorem 6.1)
For short, let us write $Z(s)$ for $Z_A(E; s)$.

First we remark that if $\mathbf{a} = (a_1, \ldots, a_d) = (0, \ldots, 0)$, then E is of the form $E(X) = uX^d$ where $u > 0$. It follows that $Z(s) = du^{1-s}\zeta(ds - d + 1)$ and Theorem 6.1 is true in this case.

One will assume in the sequel that $\mathbf{a} \neq (0, \ldots, 0)$ and set $\delta = (2 \max_j |a_j|)^{-1} > 0$. One will note in the sequel $s = \sigma + i\tau$ where $\sigma = \Re(s)$ and $\tau = \Im(s)$. It is easy to see that

$$\left| \frac{E'(n)}{E(n)^s} \right| \ll \frac{1}{n^{d\sigma - (d-1)}}.$$

It follows that $s \mapsto Z(s)$ converges absolutely in the half-plane $\{\Re(s) > 1\}$.

As the act of removing or adding a finite number of terms does not change the meromorphy or poles, we can choose the integer A as large as possible. Let us choose here $A \in \mathbb{N}^*$ such that $A \geq 2 \sup_{1 \leq j \leq d} |a_j| = \delta^{-1}$. It is clear that we can also assume without loss of generality that the polynomial E is unitary. It follows that

$$E(X) = \prod_{j=1}^d (x - a_j) \quad \text{and} \quad E'(X) = E(X) \left(\sum_{j=1}^d \frac{1}{X - a_j} \right).$$

One deduces that for all $s \in \mathbb{C}$ satisfying $\sigma = \Re(s) > 1$ there holds:

$$\begin{aligned} Z(s) &= \sum_{n=A}^{+\infty} \frac{E'(n)}{E(n)^s} = \sum_{j=1}^d \sum_{n=A}^{+\infty} \frac{1}{(n - a_j)^s \prod_{k \neq j} (n - a_k)^{s-1}} \\ &= \sum_{j=1}^d \sum_{n=A}^{+\infty} \frac{1}{n^{ds - (d-1)}} \left(1 - \frac{a_j}{n}\right)^{-s} \prod_{k \neq j} \left(1 - \frac{a_k}{n}\right)^{-s+1}. \end{aligned}$$

Let $N \in \mathbb{N}$. Lemma 6.4 and the previous relation imply that for all $s \in \mathbb{C}$ verifying $\sigma = \Re(s) > 1$ we have:

$$Z(s) = \sum_{j=1}^d \sum_{n=A}^{+\infty} \frac{1}{n^{ds-(d-1)}} \left[\sum_{\ell=0}^N c_\ell \left(f_j(s)\right) \frac{1}{n^\ell} + \frac{1}{n^{N+1}} \rho_N\left(x, f_j(s)\right) \right],$$

where $f_j(s) = (s_1, \ldots, s_d)$ with $s_k = s - 1$ if $k \neq j$ and $s_j = s$.

One deduces that for all $s \in \mathbb{C}$ verifying $\sigma = \Re(s) > 1$ there holds:

$$\begin{aligned} Z(s) &= \sum_{\ell=0}^N \left[\sum_{j=1}^d c_\ell \left(f_j(s)\right) \right] \zeta_A \left(ds - (d-1) + \ell\right) \\ &\quad + \sum_{n=A}^{+\infty} \frac{1}{n^{ds-(d-1)+N+1}} \left[\sum_{j=1}^d \rho_N \left(x; f_j(s)\right) \right], \end{aligned} \tag{29}$$

where $\zeta_A(s) := \sum_{n=A}^{+\infty} \frac{1}{n^s} = \zeta(s) - \sum_{n=1}^{A-1} \frac{1}{n^s}$.

On the other hand it is easy to see that for all $\ell \in \mathbb{N}$:

$$\begin{aligned} \sum_{j=1}^{d} c_\ell\left(f_j(s)\right) &= \sum_{j=1}^{d} (-1)^\ell \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\ |\boldsymbol{\alpha}|=\ell}} \mathbf{a}^{\boldsymbol{\alpha}} \binom{-s}{\alpha_j} \prod_{k\neq j} \binom{-s+1}{\alpha_k} \\ &= \sum_{j=1}^{d} (-1)^\ell \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\ |\boldsymbol{\alpha}|=\ell}} \mathbf{a}^{\boldsymbol{\alpha}} \frac{s+\alpha_j-1}{s-1} \prod_{k=1}^{d} \binom{-s+1}{\alpha_k} \\ &= (-1)^\ell \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\ |\boldsymbol{\alpha}|=\ell}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{k=1}^{d} \binom{-s+1}{\alpha_k} \sum_{j=1}^{d} \frac{s+\alpha_j-1}{s-1} \\ &= (-1)^\ell \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\ |\boldsymbol{\alpha}|=\ell}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{k=1}^{d} \binom{-s+1}{\alpha_k} \frac{ds-d+\ell}{s-1}. \end{aligned} \tag{30}$$

Relations (29) and (30) imply that for all $s \in \mathbb{C}$ satisfying $\sigma = \Re(s) > 1$ we have:

$$\begin{aligned} (s-1)Z(s) &= \sum_{\ell=0}^{N} \left[(-1)^\ell \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\ |\boldsymbol{\alpha}|=\ell}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{k=1}^{d} \binom{-s+1}{\alpha_k} \right] (ds-d+\ell)\,\zeta_A\left(ds-(d-1)+\ell\right) \\ &\quad +(s-1) \sum_{n=A}^{+\infty} \frac{1}{n^{ds-(d-1)+N+1}} \left[\sum_{j=1}^{d} \rho_N\left(x; f_j(s)\right) \right]. \end{aligned} \tag{31}$$

Moreover,

1. the point 2 of lemma 6.4 and the the dominated convergence theorem of Lebesgue imply that

$$s \mapsto \sum_{n=A}^{+\infty} \frac{1}{n^{ds-(d-1)+N+1}} \left[\sum_{j=1}^{d} \rho_N\left(x; f_j(s)\right) \right]$$

   is defined and is holomorphic in the half-plane $\{\sigma > 1 - \frac{N+1}{d}\}$;

2. the classical properties of the Riemann zeta function imply that the function $s \mapsto (s-1)\zeta_A(s)$ is holomorphic in the whole complex plane $\mathbb{C}$.

These last two points and identity (31) implies that $s \mapsto (s-1)Z(s)$ has a holomorphic extension to the half-plane $\{\sigma > 1 - \frac{N+1}{d}\}$. As $N \in \mathbb{N}$ is arbitrary, we deduce that $s \mapsto (s-1)Z(s)$ has a holomorphic continuation to the whole complex plane $\mathbb{C}$.
It follows that $s \mapsto Z(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with at most one possible simple pole in $s = 1$.
So to finish the proof of points 1 and 2 of Theorem 6.1, it suffices to show that $s = 1$ is a pole of residue 1.
But relation (31) with $N = 0$ implies that

$$\lim_{s\to 1} (s-1)Z(s) = \lim_{s\to 1} (ds-d)\zeta_A(ds-d+1) = 1.$$

One deduces that $s = 1$ is a simple pole of $Z(s)$ and that $Res_{s=1}Z(s) = 1$. This completes the proof of points 1 and 2 of Theorem 6.1. ■

Proof. (point 3 of Theorem 6.1)
First let us recall the classical formula

$$k\,\zeta(1-k) = (-1)^{k-1}B_k = -\Psi((X+1)^k) \quad \forall k \in \mathbb{N}^*. \tag{32}$$

Formula (32) is also valid for $k = 0$ by analytic continuation. We will use this fact in the sequel.

Let $M \in \mathbb{N}^*$. Set $N = dM$. In particular, $1 - M > 1 - \frac{N+1}{d}$. If $|\boldsymbol{\alpha}| = N + 1$, then for any $j = 1, \ldots, d$:

$$\alpha_j > M - 1 \quad \text{or} \quad \text{there exists } k \in \{1, \ldots d\} \setminus \{j\} \text{ such that } \alpha_k > M.$$

One deduces that for any $j = 1, \ldots, d$:

$$\begin{aligned} \rho_N\left(x; f_j(1-M)\right) &= (-1)^{N+1}(N+1) \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d \\ |\boldsymbol{\alpha}|=N+1}} \mathbf{a}^{\boldsymbol{\alpha}} \binom{M-1}{\alpha_j} \prod_{k\neq j}^{d} \binom{M}{\alpha_k} \\ &\quad \times \int_0^1 (1-t)^N (1-txa_j)^{M-\alpha_j} \prod_{k\neq j}^{d} (1-txa_k)^{M-1-\alpha_k}\, dt \\ &= 0. \end{aligned}$$

It follows then from (31) that

$$\begin{aligned} -MZ(1-M) &= \sum_{\ell=0}^{N} \left[(-1)^{\ell} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d \\ |\boldsymbol{\alpha}|=\ell}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{k=1}^{d} \binom{M}{\alpha_k} \right] (\ell - dM)\, \zeta_A\left(1+\ell-dM\right) \\ &= \sum_{\ell=0}^{N} \left[(-1)^{\ell} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d \\ |\boldsymbol{\alpha}|=\ell}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{k=1}^{d} \binom{M}{\alpha_k} \right] (\ell - dM)\, \zeta\left(1+\ell-dM\right) \\ &\quad - \sum_{\ell=0}^{N} \left[(-1)^{\ell} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d \\ |\boldsymbol{\alpha}|=\ell}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{k=1}^{d} \binom{M}{\alpha_k} \right] (\ell - dM) \left(\sum_{u=1}^{A-1} u^{dM-\ell-1} \right). \end{aligned} \tag{33}$$

This sum therefore splits into two parts. Remarking that if $|\boldsymbol{\alpha}| > N$ then there exists k such that $\alpha_k > M$ and hence $\binom{M}{\alpha_k} = 0$, one can compute the second part:

$$\begin{aligned} \kappa &:= -\sum_{\ell=0}^{N} \left[(-1)^{\ell} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d \\ |\boldsymbol{\alpha}|=\ell}} \mathbf{a}^{\boldsymbol{\alpha}} \prod_{k=1}^{d} \binom{M}{\alpha_k} \right] (\ell - dM) \left(\sum_{u=1}^{A-1} u^{dM-\ell-1} \right) \qquad (34) \\ &= \sum_{u=1}^{A-1} u^{dM-1} \sum_{\ell=0}^{N} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d \\ |\boldsymbol{\alpha}|=\ell}} (dM - \sum_{j=1}^{d} \alpha_j) \left[\prod_{k=1}^{d} \binom{M}{\alpha_k} \left(-\frac{a_k}{u}\right)^{\alpha_k} \right] \\ &= \sum_{u=1}^{A-1} u^{dM-1} \sum_{\boldsymbol{\alpha}\in\{0,\ldots,M\}^d} (dM - \sum_{j=1}^{d} \alpha_j) \left[\prod_{k=1}^{d} \binom{M}{\alpha_k} \left(-\frac{a_k}{u}\right)^{\alpha_k} \right]. \end{aligned}$$

Continuing this computation by splitting this sum in two, we have

$$\begin{aligned} \kappa &= dM \sum_{u=1}^{A-1} u^{dM-1} \prod_{k=1}^{d} \left(1 - \frac{a_k}{u}\right)^M - \sum_{u=1}^{A-1} \sum_{j=1}^{d} u^{dM-1} \frac{M\left(-\frac{a_j}{u}\right)}{1 - \frac{a_j}{u}} \prod_{k=1}^{d} \left(1 - \frac{a_k}{u}\right)^M \\ &= dM \sum_{u=1}^{A-1} u^{-1} E(u)^M + M \sum_{u=1}^{A-1} \sum_{j=1}^{d} \frac{a_j}{u(u-a_j)} E(u)^M \\ &= M \sum_{u=1}^{A-1} E(u)^{M-1} E'(u). \end{aligned} \tag{35}$$

Relations (32), (33), (34) and (35) imply that

$$Z(1-M)=\frac{(-1)^{dM-1}}{M}\sum_{\ell=0}^{N}\left[\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\|\boldsymbol{\alpha}|=\ell}}\mathbf{a}^{\boldsymbol{\alpha}}\prod_{k=1}^{d}\binom{M}{\alpha_k}\right]B_{dM-\ell}-\sum_{u=1}^{A-1}E(u)^{M-1}E'(u). \tag{36}$$

On the other hand, it is easy to see that

$$\begin{aligned} E(X)^M &= \prod_{j=1}^{d}(X-a_j)^M=\prod_{j=1}^{d}\left(\sum_{\alpha_j=0}^{M}\binom{M}{\alpha_j}(-a_j)^{\alpha_j}X^{M-\alpha_j}\right)\\ &= \sum_{\boldsymbol{\alpha}\in\{0,\dots,M\}^d}(-1)^{|\boldsymbol{\alpha}|}\mathbf{a}^{\boldsymbol{\alpha}}\left(\prod_{j=1}^{d}\binom{M}{\alpha_j}\right)X^{dM-|\boldsymbol{\alpha}|}=\sum_{\ell=0}^{N}(-1)^{\ell}\left[\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\|\boldsymbol{\alpha}|=\ell}}\mathbf{a}^{\boldsymbol{\alpha}}\prod_{j=1}^{d}\binom{M}{\alpha_j}\right]X^{dM-\ell}. \end{aligned}$$

Using (27), it follows that

$$\Psi(E(X+1)^M)=(-1)^{dM}\sum_{\ell=0}^{N}\left[\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}^d\\|\boldsymbol{\alpha}|=\ell}}\mathbf{a}^{\boldsymbol{\alpha}}\prod_{j=1}^{d}\binom{M}{\alpha_j}\right]B_{dM-\ell}.$$

One then deduces from (36) that $Z(1-M)=-\frac{1}{M}\Psi(E(X+1)^M)-\sum_{u=1}^{A-1}E(u)^{M-1}E'(u)$. This completes the proof of Theorem 6.1. ■

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